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## Tripartite entanglement and quantum relative entropy

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Received 4 September 2000

**Abstract.** We establish relations between tripartite pure state entanglement and additivity properties of the bipartite relative entropy of entanglement. Our results pertain to the asymptotic limit of local manipulations on a large number of copies of the state. We show that additivity of the relative entropy would imply that there are at least two inequivalent types of asymptotic tripartite entanglement. The methods used include the application of some useful lemmas that enable us to analytically calculate the relative entropy for some classes of bipartite states.

### 1. Introduction

In recent years the theory of quantum information and entanglement processing has developed rapidly. In the process our perception of entanglement has changed significantly. Entanglement used to be regarded just as a surprising manifestation of the non-locality of quantum mechanics, but today it is considered as a resource that can be exploited to implement novel quantum information processing tasks at spatially separated locations [1]. As a resource, entanglement can appear in many different forms and may not be available in the specific form necessary for the chosen task. It is therefore natural to tackle the problem of the interconversion of different forms of entanglement using local operations and classical communication only (LOCC). The local concentration of pure bipartite entanglement has already been considered in the asymptotic limit, i.e. when large numbers of entangled pairs are available [2]. In this limit it was shown that any partially entangled state can be *reversibly* converted into a smaller number of maximally entangled singlet or Einstein–Podolsky–Rosen (EPR) states. This remarkable result demonstrates that the entanglement of any pure bipartite state is essentially equivalent to that of the singlet state. One can therefore say that the set  $G_2 = \{|\text{EPR}\rangle_{AB}\}$  containing an EPR pair between systems  $A$  and  $B$  is a minimal reversible entanglement generating set (MREGS) for all bipartite pure states [3].

It is natural to ask whether there are more inequivalent forms of entanglement when one considers multi-partite pure state entanglement in the asymptotic limit; in other words, the problem is that of identifying an MREGS for multi-partite systems. Recently it has been shown that Greenberger–Horne–Zeilinger (GHZ) states are indeed inequivalent to EPR states in the asymptotic limit, i.e. there is no asymptotically reversible local procedure that allows the conversion of EPR states into GHZ states [4]. Therefore, an MREGS for tripartite systems must contain at least the GHZ state and the three possible EPRs between any two of the parties.

However, the question as to whether EPR states and GHZ states form the only kinds of tripartite entanglement, or in other words whether the set

$$G_3 = \{ |EPR\rangle_{AB}, |EPR\rangle_{AC}, |EPR\rangle_{BC}, |GHZ\rangle_{ABC} \} \quad (1)$$

is an MREGS remained unanswered in [3] and [4]. The conjecture that  $G_3$  as given in equation (1) forms an MREGS has been supported by work showing that reversible LOCC on this set yield Schmidt decomposable states [3] and also a family of states discussed in [5]. Very recently, however, Wu and Zhang [6] have shown that, without other effects<sup>†</sup>, not all four-partite states can be reversibly built using LOCC on the set of 11 maximally entangled states of two, three and four parties. Nevertheless, the structure of the MREGS for tripartite systems remains unknown.

In addition to the developments just described, some relations have been established [4] between multipartite pure state entanglement and a bipartite entanglement measure known as the *relative entropy of entanglement* [7–9]. In this paper we strengthen these relations further, obtaining new results relating the additivity of the relative entropy and the structure of the MREGS for tripartite states. In section 2 we summarize the results of [4] and present a number of useful lemmas that allow us to exploit symmetries of a quantum state to allow the analytic computation of the relative entropy of entanglement. In section 3 we assume the working hypothesis that the set  $G_3$  is an MREGS and derive a series of consequences that would follow; in particular, we show that the relative entropy of entanglement (with respect to separable states) would need to be subadditive for some classes of two-qubit states. Since to date there has been no evidence of such subadditivity for qubit states<sup>‡</sup>, in section 4 we adopt the alternative hypothesis of additivity and explore the consequences; in particular we discuss implications for the cardinality of the tripartite MREGS. In section 5 we present some final remarks.

## 2. Relative entropy, tripartite entanglement and symmetries

In this section we introduce some of the notation that we will use in the remainder of this paper. In the first section we summarize the results of [4] and in the second section we present some useful lemmas that we will employ later on.

### 2.1. Basic notation and concepts

The relative entropy of  $\rho$  with respect to any  $\sigma$  is defined as

$$S(\rho\|\sigma) := \text{tr}(\rho \log \rho) - \text{tr}(\rho \log \sigma). \quad (2)$$

This allows us to define what we mean by the *relative entropy of entanglement* and *additivity*.

**2.1.1. Relative entropy of entanglement.** For bipartite systems this entanglement measure can take three different forms,  $E_S$ ,  $E_{\text{PPT}}$  or  $E_{\text{ND}}$ <sup>§</sup>. They are defined as

$$E_X(\rho_{AB}) := \min_{\sigma_{AB} \in D(X)} S(\rho_{AB}\|\sigma_{AB})$$

<sup>†</sup> In [6], Wu and Zhang demonstrate their four-party result for the case in which the *entanglement coefficients* of [3] are positive. The case in which the entanglement coefficients are allowed to be negative may have an interpretation involving catalysis [3].

<sup>‡</sup> It has recently come to our attention that there are some two-qutrit states for which the relative entropy of entanglement with respect to both separable and positive partial transpose (PPT) states is subadditive [10]. However, to date, all analytic and numerical computations seem to support additivity for two-qubit states.

<sup>§</sup> The natural extensions to multi-partite entanglement have been presented in [11].

where  $X = S, PPT, ND$  and the minimum is taken over the set  $D$  of separable (S), non-distillable (ND) or positive partial transpose (PPT) density matrices [7–9, 12]. These measures can further be ‘regularized’ for use in discussions involving asymptotic manipulations:

$$E_X^{\text{reg}}(\rho_{AB}) := \lim_{n \rightarrow \infty} (1/n) \min_{\sigma_{AB} \in D(X)} S(\rho_{AB}^{\otimes n} \| \sigma_{AB}).$$

It is important to note that in the case where  $\rho$  is either a pure state or a separable state then all the measures are equal:  $E_S(\rho) = E_{PPT}(\rho) = E_{ND}(\rho) = E_S^{\text{reg}}(\rho) = E_{PPT}^{\text{reg}}(\rho) = E_{ND}^{\text{reg}}(\rho)$ .

2.1.2. *Additivity.* There are two major types of additivity which will concern us in this paper.

(a) If an entanglement measure  $E$  satisfies  $E^{\text{reg}}(\rho) = E(\rho)$  we will say that  $E$  is an *asymptotically additive* measure.

(b) If an entanglement measure  $E$  satisfies for all  $\rho_1, \rho_2$  the relation  $E(\rho_1 \otimes \rho_2) = E(\rho_1) + E(\rho_2)$  then we will say that  $E$  is a *fully additive* measure.

The connection between the relative entropy of entanglement and multipartite entanglement was first pointed out in [4], where it was shown that if two multipartite pure states can be reversibly interconverted then the relative entropy of entanglement must remain constant for any two parties  $i, j$ . This remarkable result can be used to derive constraints that must hold if the set  $G_3$  is an MREGS for tripartite pure states. In particular, suppose that we reversibly and asymptotically wish to create a tripartite pure state  $|\Psi_{ABC}\rangle$  between parties  $A, B$  and  $C$ , and that per output copy of  $|\Psi_{ABC}\rangle$  we will use  $g$  GHZ states and  $s_{ij}$  EPR pairs between parties  $i$  and  $j$ . Then, denoting the reduced density matrices of parties  $i, j$  by  $\rho_{ij}$ , we find that the following relationships must hold:

$$E_X^{\text{reg}}(\rho_{ij}) = s_{ij} \tag{3}$$

$$S(\rho_A) = g + s_{AB} + s_{AC} \tag{4}$$

$$S(\rho_B) = g + s_{AB} + s_{BC} \tag{5}$$

$$S(\rho_C) = g + s_{AC} + s_{BC} \tag{6}$$

where  $S(\rho_i)$  represents the Von Neumann entropy of the reduced density matrix of party  $i$ †.

It is an open question whether all tripartite states satisfy the equations (3)–(6). Any counterexample would be a state which cannot be generated reversibly from the set  $G_3$ , representing a new kind of asymptotic tripartite entanglement. Unfortunately there are no known general techniques for calculating  $E_X^{\text{reg}}(\rho)$ . Despite this, we were able to obtain some progress in establishing relations between additivity questions and the structure of the MREGS for tripartite pure states. In particular, we present classes of states which are potential candidates for violating relations (3)–(6).

2.2. *Symmetries and continuity*

In this subsection we prove a number of useful lemmas that simplify the computation of the relative entropy of entanglement for states that possess symmetries. In addition, we state a lemma due to Donald and Horodecki concerning the continuity of the relative entropy of entanglement.

We begin by recalling a lemma by Rains [9] which enables us to use local symmetries of the state  $\rho_{AB}$  to narrow down the possible set of optimal states. Then we extend this lemma to non-local symmetry operations.

† In [4] relationships similar to (3) were derived using  $E_S(\rho_{ij})$  in place of  $E_S^{\text{reg}}(\rho_{ij})$ . Unfortunately this requires the implicit assumption that  $E_S(\rho_{ij})$  is asymptotically additive, a result which has not yet been established.

**Lemma 1 ([9]).** *If a bipartite density matrix is invariant under a sub-group of local unitary transformations, then the optimal PPT state can also be chosen to be invariant under the same sub-group.*

Although the proof can be found in [9] we present it here to clarify how this theorem can be generalized to non-local symmetry groups.

**Proof.** Let there be a bipartite density matrix  $\rho$  which is invariant under a sub-group of local transformations  $G = \{U_i \otimes V_i\}$ , with an optimal PPT state  $\sigma$ . For simplicity, let us assume that the group is discrete (the generalization to continuous groups is straightforward [9]). Then  $E_S(\rho)$  is given by

$$E_S(\rho) = S(\rho\|\sigma) = S(\rho\|U_i \otimes V_i \sigma U_i^\dagger \otimes V_i^\dagger) \quad (7)$$

due to the invariance of the relative entropy under unitary transformations and the invariance of  $\rho$  under  $G$ . We can then define another state  $\sigma_s$  such that

$$\sigma_s = \frac{1}{|G|} \sum_{i=1}^{|G|} U_i \otimes V_i \sigma U_i^\dagger \otimes V_i^\dagger. \quad (8)$$

It is important to note that  $\sigma_s$  is by construction both PPT and invariant under any unitary transformation selected from  $G$  (due to the rearrangement theorem [13]).

The convexity of the relative entropy [14] hence implies that

$$S(\rho\|\sigma_s) \leq \frac{1}{|G|} \sum_{i=1}^{|G|} S(\rho\|U_i \otimes V_i \sigma U_i^\dagger \otimes V_i^\dagger) = S(\rho\|\sigma). \quad (9)$$

As  $\sigma$  was already an optimal PPT state, this equation must in fact be a strict equality. Hence  $\sigma_s$  is also an optimal PPT state for  $\rho$ , and by construction is invariant under the same group  $G$ .  $\square$

**Corollary 1.** *This lemma can in fact be extended to include symmetry groups which include non-local operations and even operations which are non-physical (such as transposition). Suppose that there is a set of operations  $\{\Lambda\}$  of the symmetry group which is either non-local or non-physical, but still takes density matrices to density matrices. Then the above reasoning still applies as long as an optimal PPT state  $\sigma$  exists for which  $S(\rho\|\Lambda\sigma) = S(\rho\|\sigma)$  and  $\Lambda(\sigma)$  is still PPT.*

**Lemma 2.** *Any disentangled state  $\sigma$  which is optimal for a negative partial transpose (NPT) state  $\rho$  must give a partial transposed state  $\sigma^\Gamma$  with at least one zero eigenvalue.*

**Proof.** Consider an NPT state  $\rho$  for which the optimal PPT state is  $\sigma$ . Then the convexity of the relative entropy implies that for all  $p \in [0, 1)$

$$S(\rho\|p\sigma + (1-p)\rho) \leq pS(\rho\|\sigma) + (1-p)S(\rho\|\rho) < S(\rho\|\sigma). \quad (10)$$

This means that  $p\sigma + (1-p)\rho$  must necessarily be an NPT state  $\forall p \in [0, 1)$ , otherwise  $\sigma$  would strictly not be an optimal PPT state. Now if all the eigenvalues of  $\sigma^\Gamma$  are non-zero, then it would be possible to mix with  $\sigma$  a small amount of  $\rho$  and still keep the resulting density matrix PPT. Hence at least one of the eigenvalues of  $\sigma^\Gamma$  must be zero.  $\square$

Finally we state a lemma concerning the continuity of the relative entropy of entanglement which is due to Donald and Horodecki [15].

**Lemma 3.**  *$E_S(\rho)$  is continuous. Denoting  $\text{tr}(|\rho_1 - \rho_2|)$  by  $\Delta$  we have the inequality*

$$|E_S(\rho_1) - E_S(\rho_2)| \leq 2 \log(\dim(\mathcal{H}_{AB}))\Delta - 2\Delta \log(\Delta) + 4\Delta \quad (11)$$

where  $\mathcal{H}_{AB}$  is the Hilbert space of  $\rho_i$ . The proof is given by Donald and Horodecki [15].

### 3. Consequences if the set $G_3$ is an MREGS

In this section we discuss some results which must hold if the set  $G_3$  turns out to be an MREGS for tripartite states. We start in section 3.1 by proving that  $E_S$  must be asymptotically subadditive for a class of two-qubit states if  $G_3$  is an MREGS. Then in section 3.2 we present implications that set  $G_3$  being an MREGS would have for full additivity of entanglement measures, and in section 3.3 we comment on the possibility of obtaining analytic expressions for  $E_X^{\text{reg}}$  for some classes of states.

#### 3.1. Asymptotic subadditivity of $E_S$

**Theorem 1.** *If the set  $G_3$  is an MREGS for all tripartite pure states then  $E_S(\rho)$  must be asymptotically subadditive for some two-qubit states.*

In order to prove this theorem we will show that there are states of the form

$$|\psi\rangle = e|000\rangle + f|101\rangle + f|110\rangle \tag{12}$$

for which the non-regularized relative entropies  $E_{\text{PPT}}$  and  $E_S$  do not satisfy modified versions of equations (4)–(6), with

$$E_X(\rho_{ij}) = s_{ij} \tag{13}$$

instead of the regularized relative entropies. This will allow us to draw the conclusion that for the set  $G_3$  to be an MREGS for tripartite pure states we need  $E_S$  to be asymptotically subadditive for two-qubit states [10]. The proof of theorem 1 uses the lemmas that have been proven in section 2.2, which help to calculate  $E_S$  using the symmetries of the quantum state equation (12). We compute the reduced density operators of all subsystems to find (setting  $e$  and  $f$  real)

$$\rho_{AB}(e^2, f^2) = \rho_{AC}(e^2, f^2) = \begin{pmatrix} e^2 & 0 & 0 & ef \\ 0 & 0 & 0 & 0 \\ 0 & 0 & f^2 & 0 \\ ef & 0 & 0 & f^2 \end{pmatrix} \tag{14}$$

$$\rho_{BC}(e^2, f^2) = \begin{pmatrix} e^2 & 0 & 0 & 0 \\ 0 & f^2 & f^2 & 0 \\ 0 & f^2 & f^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{15}$$

Note that if  $e$  and  $f$  are non-zero then both these states have NPT, which is equivalent to inseparability for two-qubit states [16, 17]. In the following we will only be discussing two-qubit states, and hence we will use the terms separable and PPT interchangeably.

We begin by making the assumption that  $E_S$  is asymptotically additive for all two-qubit states. Then from (4)–(6) and the fact that  $\rho_{AB} = \rho_{AC}$  it follows that a necessary condition for  $|\psi\rangle$  to be obtainable from set  $G_3$  by reversible LOCC is given by

$$E_S(\rho_{AB}) + S(\rho_{AB}) = E_S(\rho_{BC}) + S(\rho_{BC}) \tag{16}$$

where  $S(\rho)$  is the von Neumann entropy of the state  $\rho$ . Example 1 of [7] gives us that

$$E_S(\rho_{BC}) = 2(f^2 - 1) \log_2(1 - f^2) + (1 - 2f^2) \log_2(1 - 2f^2) \tag{17}$$

and by direct computation we find that

$$S(\rho_{BC}) = -(1 - 2f^2) \log_2(1 - 2f^2) - 2f^2 \log_2(2f^2) \tag{18}$$

$$S(\rho_{AB}) = -f^2 \log_2 f^2 - (1 - f^2) \log_2(1 - f^2). \tag{19}$$

Combining these equations with equation (16), we see that if  $E_S$  is asymptotically additive for  $\rho_{AB}$  and  $\rho_{BC}$ , then  $G_3$  can only be an MREGS for states of the form (12) if

$$E_S(\rho_{AB}) = (f^2 - 1) \log_2(1 - f^2) - 2f^2 \log_2(2f^2) + f^2 \log_2(f^2). \quad (20)$$

In the following we are going to bound  $E_S(\rho_{AB})$  analytically and show that equation (20) is violated, thereby proving theorem 1.

Essentially our task is to constrain the closest (optimal) PPT state  $\sigma_{AB}$  to the density operator  $\rho_{AB}$ . We will accomplish this using the lemmas of section 2.2. We sequentially apply these lemmas in order to bound  $E_S(\rho_{AB})$ . We will use symmetry arguments first. The elements of  $\rho_{AB}$  are real, and therefore invariant under the act of transposition or complex conjugation in the computational basis. Transposition has two properties which allow the application of corollary 1. The first is that transposition in a product basis does not change the PPT properties of a state. The second is that as  $\rho_{AB}$  is symmetric,  $-\text{tr}\{\rho \log(\sigma)\}$  and hence  $S(\rho\|\sigma)$  are also invariant under transposition of  $\sigma$ . Therefore we can utilize the above corollary and demand that our optimal PPT state be symmetric, and hence also real. Furthermore, the state  $\rho_{AB}$  is invariant under the local group

$$\mathcal{X} = \{I, \sigma_z \otimes \sigma_z\}$$

and the non-local group

$$\mathcal{Y} = \{I, W\}$$

where  $W = (|00\rangle\langle 00| + |01\rangle\langle 01| - |10\rangle\langle 10| + |11\rangle\langle 11|)$ . Let us first consider the action of  $\mathcal{X}$ . Having fixed the elements to be real, it is a straightforward calculation to show that the only density matrices which are invariant under  $\mathcal{X}$  must be of the form

$$\sigma_{AB} = \begin{pmatrix} x & 0 & 0 & v \\ 0 & y & w & 0 \\ 0 & w & z & 0 \\ v & 0 & 0 & u \end{pmatrix}. \quad (21)$$

Restricting our attention to such states, we can now consider the action of the non-local symmetry group  $\mathcal{Y}$ . Applying the element  $W$  of  $\mathcal{Y}$  to these states is equivalent to changing  $w$  to  $-w$ . One can easily see that this transformation does not change the PPT properties of these states. Therefore, utilizing the above corollary we can also require that our optimal state be invariant under changing  $w$  to  $-w$ , in which case we can set  $w = 0$ . Now we are in a position to apply lemma 2 to  $\sigma_{AB}$  of equation (21). Having already set  $w = 0$  we now require  $\sigma_{AB}^\Gamma$  to have at least one zero eigenvalue. If  $x$  or  $u$  are zero then  $v$  must be zero to maintain positivity of  $\sigma_{AB}$ . However, this cannot be the case if the elements  $e^2$  and  $f^2$  of  $\rho_{AB}$  are both non-zero, as then  $E_S(\rho_{AB})$  becomes infinite. We will only consider this case, as otherwise the state in equation (12) trivially satisfies equation (16). Therefore we can set  $x, u$  to be non-zero, and the only way that  $\sigma_{AB}^\Gamma$  can have at least one zero eigenvalue and still be PPT is if  $v = \sqrt{yz}$ . Hence the optimal state can be made to take the form

$$\sigma_{AB} = \begin{pmatrix} x & 0 & 0 & \sqrt{yz} \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ \sqrt{yz} & 0 & 0 & 1 - x - y - z \end{pmatrix}. \quad (22)$$

To optimize the relative entropy of entanglement with just the three real parameters left we have to solve the following partial differential equations:

$$\frac{\partial}{\partial k} \{-\text{Tr}(\rho_{AB} \log_2 \sigma_{AB})\} = 0 \quad k = x, y, z. \quad (23)$$

In general this can only be done numerically as the resulting equations are nonlinear in the parameters of  $\sigma$ . However, as we are merely looking for an example for which equation (20) does not hold, we are free to set convenient values for the parameters of  $\sigma_{AB}$  and then *analytically* solve equations linear in the parameters of  $\rho_{AB}$ . A problem with this technique is that it does not guarantee that the resulting  $\rho_{AB}$  takes the form of (14). In fact, any two-party subsystem of a pure  $2 \times 2 \times 2$  state must be rank-2, whereas generic calculations utilizing this technique tend to result in  $\rho_{AB}$  of higher rank. A likely explanation for this is the measure zero of rank-two states in the Hilbert space of two qubits. Nevertheless, it is possible to approach extremely closely a state of the form of equation (14) using this technique. We analytically obtained one particular example for  $\sigma$  given by  $x = 0.487\,547\,3233$ ,  $y = 0.128\,640\,6856$  and  $z = 0.295\,307\,3521$ . Unfortunately, the analytical expressions for the parameters of the NPT state derived in this way are extremely long, so for brevity we only write the matrix elements and values in subsequent calculations to 12 significant digits<sup>†</sup>:

$$\rho_{AB}^a = \rho_{AB}(2/3, 1/6) + 10^{-10} \begin{pmatrix} 0.672 & 0 & 0 & 1.32 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1.67 & 0 \\ 1.32 & 0 & 0 & 0.995 \end{pmatrix}$$

where  $\rho_{AB}(2/3, 1/6)$  is defined via equation (14). This results in a relative entropy

$$E_S(\rho_{AB}^a) = 0.354\,761\,489\,848. \quad (24)$$

Note that the state  $\rho_{AB}^a$  is very close to the state  $\rho_{AB}(2/3, 1/6)$ . Using lemma 3 one can hence bound the entanglement of  $\rho_{AB}(2/3, 1/6)$  as

$$E_S[\rho_{AB}(2/3, 1/6)] = 0.354\,761\,489\,848 \pm 3.1 \times 10^{-8}. \quad (25)$$

This value is not compatible with the prediction of  $E = 0.3167$ , obtained from equation (20). This means that the value of the non-regularized entanglement  $E_S(\rho_{AB})$  is too high to satisfy equation (20), and hence the set  $G_3$  can only be an MREGS for states of three qubits if  $E_S$  is asymptotically subadditive, thus finishing the proof of theorem 1.

The result above showed that provided  $G_3$  is an MREGS, then  $\rho_{AB}$  given by (12) with  $e^2 = 2/3$  and  $f^2 = 1/6$  must be asymptotically subadditive. It is clear from the method we used that similar proofs can be made for different values of  $e$  and  $f$ . We have written a program that calculates  $E_S$  using a gradient search algorithm [7] to test this hypothesis. The results we obtained suggest that, given the assumption that  $G_3$  is an MREGS,  $\rho_{AB}$  must be asymptotically subadditive for generic values of  $e$  and  $f$ . Equations (3)–(6) are automatically satisfied when  $e = f$  because of the symmetry under permutations of the three parties. Using our program we have numerically checked the additivity of  $E_S$  for two copies of the state  $\rho_{AB}$  and have found that, within the limits of numerical precision, additivity is satisfied in this case. This provides some weak evidence that indeed  $E_S$  is additive for this class of states and that therefore  $G_3$  is not an MREGS. The rigorous proof of this result has, however, not been completed so far.

### 3.2. Full additivity of $E_S^{\text{reg}}$

Although we have shown that  $E_S$  must be asymptotically subadditive for  $G_3$  to be an MREGS, Wu and Zhang recently made a stronger claim that full additivity of  $E_S$  is required [6]. Their original discussion, however, was based on equations (23) of [4], and hence also relies on the implicit assumption of asymptotic subadditivity present in [4]. Here we present a corrected version of their result.

<sup>†</sup> The full analytic calculations can be obtained in the form of a *Mathematica* worksheet by e-mailing any of the authors.



**Theorem 2.** *If the set  $G_3$  is an MREGS for tripartite pure states then  $E_X^{\text{reg}}$  must be fully additive.*

**Proof.** The proof is as given in [6], except replacing  $E_X$  with  $E_X^{\text{reg}}$ .  $\square$

Thus the question of  $G_3$  being an MREGS can be related to both the asymptotic subadditivity of  $E_S$  and the full additivity of  $E_X^{\text{reg}}$ . In fact even stronger statements can be made about the relationship between the regularized forms of the relative entropy of entanglement.

**Theorem 3.** *If  $G_3$  is an MREGS then  $E_S^{\text{reg}} = E_{\text{PPT}}^{\text{reg}} = E_{\text{ND}}^{\text{reg}}$ .*

**Proof.** The proof relies on the fact that for pure states and separable states all of these measures are identical [7, 9, 11, 12]—they are equal to the entropy of the reduced density matrix in the case of bipartite pure states, and are zero for all separable states. Therefore the right-hand sides of conditions (3)–(6) are in fact the same for all of these measures. Hence the left-hand sides of these equations must also be the same, and all regularized versions are equivalent if the set  $G_3$  is an MREGS.  $\square$

Theorem 3 has interesting consequences. Although theorem 1 above gives an example of a  $2 \times 2 \times 2$ -dimensional (three-qubit) state which cannot be obtained by reversible LOCC on set  $G_3$  if  $E_S$  is asymptotically additive, theorem 3 can in fact be used to give higher-dimensional states with the same property. All PPT bound [18] entangled states (which only exist in dimensions of at least two qutrits) have  $E_{\text{PPT}}^{\text{reg}} = 0$  and  $E_S \neq 0$ . Hence any purifications of bound entangled states would only be reversibly obtainable from  $G_3$  if  $E_S$  is asymptotically subadditive for all bound entangled states.

### 3.3. Calculating $E_X^{\text{reg}}$

If it is true that  $G_3$  is an MREGS for tripartite pure states, then we can use relations (3)–(6) to obtain  $E_X^{\text{reg}}$  for some bipartite states. A simple example is obtained from the tripartite state

$$|\Lambda(a, b)\rangle = a|000\rangle + b|100\rangle + b|101\rangle + b|110\rangle + b|111\rangle \quad (26)$$

whose reduced density matrix for parties  $A$  and  $B$  is

$$\rho_{AB} = \rho_{AC} = \begin{pmatrix} a^2 & 0 & ab & ab \\ 0 & 0 & 0 & 0 \\ ab & 0 & 2b^2 & 2b^2 \\ ab & 0 & 2b^2 & 2b^2 \end{pmatrix}. \quad (27)$$

It is easy to show that  $\rho_{BC}$  is separable (it is PPT). Since  $\rho_{AB} = \rho_{AC}$ , the relations (3)–(6) mean that if  $G_3$  is an MREGS, then it must be true that

$$E_S^{\text{reg}}(\rho_{AB}) = S(\rho_{BC}) - S(\rho_{AB}). \quad (28)$$

A closed formula for  $E_S^{\text{reg}}(\rho_{AB})$  can then be obtained by directly computing the von Neumann entropies from equations (26)–(28). Similar calculations can be used to find  $E_X^{\text{reg}}$  for many states, provided they are obtainable from  $G_3$  by LOCC.

## 4. Consequences if $E_s$ is asymptotically additive

In section 3 we discussed various consequences that would follow if set  $G_3$  can be proven to be an MREGS for tripartite pure states. In particular, we have shown that this would entail subadditivity of the relative entropy of entanglement  $E_S$ . Despite extensive numerical work [7, 19], no indication of subadditivity of  $E_S$  has been found to date for two-qubit states [10].

In this section we analyse what would follow if  $E_S$  turned out to be asymptotically additive for two-qubit states.

The first and most obvious conclusion would be that the set  $G_3$  could not be an MREGS as follows directly from theorem 1 for three-qubit states and theorem 3 for higher-dimensional states. It is interesting to note that the states discussed in theorem 1 are examples of the ‘W’-states of [20], which have been shown to be inequivalent to GHZ states in the sense that one class of states cannot be converted to the other with non-zero probability (for single-copy manipulations). If  $E_S$  is asymptotically additive, it follows from theorem 1 that a similar inequivalence persists in the asymptotic and reversible case.

There are also implications for the minimum size which the MREGS must have. We have performed numerical tests which indicate that the tripartite states

$$|\Lambda(a, b)\rangle = a|000\rangle + b|100\rangle + b|101\rangle + b|110\rangle + b|111\rangle \quad (29)$$

do not satisfy equations (3)–(6) with the non-regularized relative entropies (w.r.t. separable states). Note that for this state  $\rho_{BC}$  can easily be shown to be separable (it is invariant under partial transposition), whereas the states of  $AB$  and  $AC$  are inseparable—hence party  $A$  is the ‘odd one out’. This implies that, should the relative entropies for the reduced density matrices of this state be additive, we would require that at least one more state be added to the set  $G_3$  to turn it into an MREGS. Moreover, this state would have to be separable across parties  $B$  and  $C$ . However, if one such state were added, due to symmetry we would in fact require a further two states to be added as well—corresponding to making the other parties  $B$  or  $C$  the ‘odd ones out’ instead of party  $A$ . This would mean that the minimum cardinality of a tripartite MREGS would have to be seven—the GHZ, three EPR pairs and the three above kinds of state<sup>†</sup>. A similar argument cannot be applied for the state of theorem 1 as it is not separable between any two parties.

## 5. Conclusion

Our results strengthen the connection between the bipartite relative entropy of entanglement and the structure of the MREGS for tripartite pure states. In particular, we have shown that in the case where  $E_S$  turns out to be asymptotically additive for two-qubit states, then symmetry arguments can be used to show that the states we have investigated cannot be obtained reversibly from the set  $G_3 = \{|\text{EPR}\rangle_{AB}, |\text{EPR}\rangle_{AC}, |\text{EPR}\rangle_{BC}, |\text{GHZ}\rangle_{ABC}\}$ . On the other hand, if  $E_{\text{reg}}^X$  is *not* fully additive, then  $G_3$  also cannot be an MREGS. It would be interesting to further investigate the relationship between bipartite mixed state entanglement and pure tripartite entanglement, as results in one area may bear fruit in the other.

## Acknowledgments

We would like to thank Jens Eisert, Leah Henderson and Lucien Hardy for valuable discussions. We also thank Matthew Donald for comments on  $E_S^{\text{reg}}$ , and are especially grateful to Max Sacchi for pointing out some serious mistakes in earlier drafts of this work. We would also like to thank Reinhard Werner for communicating the results of [10] prior to publication. We acknowledge support from the UK ORS Awards scheme, the Brazilian agency Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES), the UK Engineering and Physical

<sup>†</sup> It might be argued that the GHZ state may be replaced by some other tripartite state which together with EPR pairs could then be used to generate GHZs and the  $|\Lambda\rangle$  states. However, in order to be able to generate the GHZ in the case where  $E_S$  is asymptotically additive, any such state would also have to be separable across any two parties, in which case it could not be used to generate the  $|\Lambda\rangle$  states.

Sciences Research Council (EPSRC), the Leverhulme Trust and the European Union project EQUIP.

## References

- [1] Plenio M B and Vedral V 1998 *Contemp. Phys.* **39** 431
- [2] Bennett C H, Bernstein H J, Popescu S and Schumacher B 1996 *Phys. Rev. A* **53** 2046
- [3] Bennett C H, Popescu S, Rohrlich D, Smolin J A and Thapliyal A V 1999 *Los Alamos Preprint* quant-ph/9908073 (*Phys. Rev. A* at press)
- [4] Linden N, Popescu S, Schumacher B and Westmoreland M 1999 *Los Alamos Preprint* quant-ph/9912039
- [5] Vidal G, Dür W and Cirac J I 2000 *Phys. Rev. Lett.* **85** 658
- [6] Wu S and Zhang Y 2000 *Los Alamos Preprint* quant-ph/0004020
- [7] Vedral V and Plenio M B 1998 *Phys. Rev. A* **57** 1619
- [8] Plenio M B, Virmani S and Papadopoulos P 2000 *J. Phys. A: Math. Gen.* **33** L193  
Vedral V, Plenio M B, Rippin M A and Knight P L 1997 *Phys. Rev. Lett.* **78** 2275
- [9] Rains E M 1999 *Phys. Rev. A* **60** 179
- [10] Vollbrecht K G H and Werner R F 2000 *Los Alamos Preprint* quant-ph/0010095
- [11] Vedral V, Plenio M B, Jacobs K A and Knight P L 1997 *Phys. Rev. A* **56** 4452
- [12] Vedral V 1999 *Phys. Lett. A* **262** 121
- [13] Tinkham M J 1964 *Group Theory and Quantum Mechanics* (New York: McGraw-Hill)
- [14] Wehrl A 1978 *Rev. Mod. Phys.* **50** 221
- [15] Donald M J and Horodecki M 1999 *Phys. Lett. A* **264** 257
- [16] Peres A 1996 *Phys. Rev. Lett.* **77** 1413
- [17] Horodecki M, Horodecki P and Horodecki R 1997 *Phys. Rev. Lett.* **78** 574
- [18] Horodecki M, Horodecki P and Horodecki R 2000 *Phys. Rev. Lett.* **84** 4260
- [19] Horodecki P, Smolin J A, Terhal B M and Thapliyal A V 1999 *Los Alamos Preprint* quant-ph/9910122
- [20] Dür W, Vidal G and Cirac J I 2000 *Los Alamos Preprint* quant-ph/0005115 (*Phys. Rev. A* at press)